Exercise 19: Consecutive digits

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Inhaltsverzeichnis

Collatz sequence

Collatz Problem

The Collatz function is defined as

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ 3x + 1 & \text{if } x \text{ is odd} \end{cases}$$

The Collatz sequence uses this function recursively, until $f(c_n) = 1$

$$c_{n+1} = f(c_n)$$

starting with $c_0 = 13$ we generate the following sequence:

$c_{0,1,2,\ldots} = 13, 40, 20, 10, 5, 16, 8, 4, 2, 1$

in Pseudocode

```
function COLLATZ(n)
    output n
    if n < 2 then                         ▷ termination condition
        return
    end if
    if n mod 2 = 0 then                   ▷ even or odd
        COLLATZ(n/2)                      ▷ recursion
    else
        COLLATZ(3n + 1)
    end if
end function
```

in Pseudocode

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function COLLATZ(n)
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in Pseudocode

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function COLLATZ(n)
    output n
    if n < 2 then                              ▷ termination condition
        return
    end if
    if n mod 2 = 0 then                        ▷ even or odd
        COLLATZ(n/2)                           ▷ recursion
    else
        COLLATZ(3n + 1)
    end if
end function
```

in R code

```
collatz <- function(n)
{
        print(n)
        if(n<2) return(FALSE)
        if(n %% 2==0)
                collatz(n/2)
        else
                collatz(3*n+1)
}
```

in R code (returning a vector)

```
collatz <- function(n)
{

        if(n<2) return(1)
        if(n %% 2==0)
                return(c(n, collatz(n/2)))
        else
                return(c(n, collatz(3*n+1)))
}
```

Proof of Collatz Problem

Proof of Collatz Problem I

We prove the Collatz conjecture, by trying to complete an incomplete sequence, and eliminating out the contradictions. We also check and eliminate out the possibility of an infinite loop. In the end we show that every integer has a finite stopping time, which in other words solves the Collatz conjecture.
Consider the function

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ 3x + 1 & \text{if } x \text{ is odd} \end{cases}$$

The Collatz conjecture states that there exists an integer $d(\geq 0)$ corresponding to x such that $f^d(x) = 1$ for all natural numbers x where $f^d(x) = f(f(f(\ldots\text{d times}\ldots(f(f(f(x)))))))$

Proof of Collatz Problem II

Considering only odd integers: Now if we consider only odd integers of the Collatz sequence, and modify the function as follows

$$T(x) = \frac{(3x+1)}{2^k}$$

where k is the highest power of 2 that divides $3x + 1$
If we can prove that for all odd integers the stopping time corresponding to T(x) is finite, then the Collatz problem is proved.
Consider a given incomplete sequence $a_1, a_2, \ldots, a_m$, where $a_k = T^k(a_1)$, and a_1 is the minimum among $a_1, \ldots, a_m$. Then we shall have $2^{S_m} = (3 + \frac{1}{a_1})(3 + \frac{1}{a_1})\ldots(3 + \frac{1}{a_1})\frac{a_1}{a_{m+1}}$
But since a_1 is mimimum, hence, we shall,

$$2^{S_m} < (3 + \frac{1}{a_1})^m \frac{a_1}{a_{m+1}} \qquad (1)$$

Proof of Collatz Problem III

Now lets suppose

$$a_{m+1} < a_1 \tag{2}$$

$$\frac{3^m}{2^{S_m}}a_1 + \frac{c_m}{2^{S_m}} < a_1 \tag{3}$$

Now, for $m > 3, c_m > 3^m$, hence we have

$$\frac{3^m}{2^{S_m}}a_1 + \frac{3^m}{2^{S_m}} < a_1 \tag{4}$$

$$\frac{3^m}{2^{S_m}}(a_1 + 1) < a_1 \tag{5}$$

$$3^m(\frac{1 + a_1}{a_1}) < 2^{S_m} \tag{6}$$

Proof of Collatz Problem IV

From (1) and (6) we have

$$3^m(\frac{1+a_1}{a_1}) < (3+\frac{1}{a_1})^m \frac{a_1}{a_{m+1}} \tag{7}$$

From (7) we get after rearranging the terms

$$3^m(a_1^2 - a_1 a_{m+1} - a_{m+1}) +^m C_1 3^{m-1} a_1 + ... + \frac{{}^m C_m}{a_1^{m-2}} > 0 \tag{8}$$

Here by (2) we have the equation

$$L.H.S > 3^m(-a_{m+1}) +^m C_1 3^{m-1} a_1 + ... + \frac{{}^m C_m}{a_1^{m-2}} \tag{9}$$

Proof of Collatz Problem V

Which reduces to the equation (10) if we assume (11) to be true

$$(ma_1 - 3a_{m+1}) + \frac{2^m - m - 1}{3^{m-1}a_1^{m-2}} > 0 \tag{10}$$

$$3^m(-a_{m+1}) + {}^mC_1 3^{m-1}a_1 + \ldots + \frac{{}^mC_m}{a_1^{m-2}} > (ma_1 - 3a_{m+1}) + \frac{2^m - m - 1}{3^{m-1}a_1^{m-2}} > 0 \tag{11}$$

Now for $m > 3$

$$\frac{2^m - m - 1}{3^{m-1}a_1^{m-2}} < 1 \tag{12}$$

Hence,

$$(ma_1 - 3a_{m+1}) > 0 \tag{13}$$

which is true since $m > 3$ and $a_1 > a_{m+1}$

Proof of Collatz Problem VI

Statement 1 Hence, we get from (1) and (2) and assumption (11), the equation (12) which is true. If (1), (2) or (11) was false, (12) would also have been false. Hence, (2) as well as (12) is true. So we can state that, for any arbitrary number a_1 there exists $m > 3$ such that $a_{m+1} < a_1$. Which , obviously, gives a hint that all integers have a finite stopping time.

Now consider, if for all $m \geq 1$

$$a_{m+1} > a_1 \tag{14}$$

$$\frac{3^m}{2^{S_m}} a_1 + \frac{c_m}{2^{S_m}} > a_1 \tag{15}$$

But, for $m \geq 1$

$$\frac{3^m}{2^{S_m}} a_1 + \frac{m4^{m-1}}{2^{S_m}} \geq \frac{3^m}{2^{S_m}} a_1 + \frac{c_m}{2^{S_m}}, \tag{16}$$

Proof of Collatz Problem VII

Therefore,

$$\frac{3^m}{2^{S_m}}a_1 + \frac{m4^{m-1}}{2^{S_m}} > a_1 \tag{17}$$

Implies,

$$a_1 < \frac{m4^{m-1}}{2^{S_m} - 3^m} \tag{18}$$

Which in turn, implies,

$$a_1 < m4^{m-1} \tag{19}$$

Since $2^{S_m} - 3^m \geq 1$ for all $m \geq 1$
But here, a_1 is arbitraily chosen, and doesn't depend on the value of m Hence, (19) does not hold true for all m. So, our assumption $a_{m+1} > a_m$ for all $m \geq 1$, must be wrong

Proof of Collatz Problem VIII

This gives us an indication that any infinite trajectory cannot have a minimum a_1. Which is clearly impossible. Hence, infinite trajectories do not exist. Considering the fact that infinite trajectories do not exist, we may say from **Statement 1** that $m > 3$ has a finite value. All of the above sums up to the fact, that every number has a finite stopping time.
q.e.d.

... just kidding ☺

This was just a load of crap, the Collatz Problem remains unproven.